

# INTERSECTIONS OF TRANSLATED ALGEBRAIC SUBTORI

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**ABSTRACT.** We exploit the classical correspondence between finitely generated abelian groups and abelian complex algebraic reductive groups to study the intersection theory of translated subgroups in an abelian complex algebraic reductive group, with special emphasis on intersections of (torsion) translated subtori in an algebraic torus.

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## 1. INTRODUCTION

**1.1. Motivation.** In this note, we study the intersection theory of translated subtori in a complex algebraic torus, and, more generally, of translated subgroups in an abelian complex algebraic reductive group. The motivation for this study comes in large part from the investigation of characteristic varieties and homological finiteness properties of abelian covers, embarked upon in [10] and [11].

As shown by Arapura [1], the jump loci for cohomology with coefficients in rank 1 local systems on a connected, smooth, quasi-projective variety  $X$  consist of translated subtori of the character torus of  $\pi_1(X)$ . Understanding the way these subtori intersect gives valuable information on the Betti numbers of regular, abelian covers of  $X$ , see for instance [10, 11].

Studying the intersection theory of arbitrary subvarieties in a complex algebraic torus is beyond the scope of this work. Nevertheless, the torsion points of the intersection of such subvarieties can be located by considering the intersection of suitable translated

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subtori. Indeed, as shown by M. Laurent in [4], given any subvariety  $V \subset (\mathbb{C}^*)^r$ , there exist torsion-translated subtori  $Q_1, \dots, Q_s$  in  $(\mathbb{C}^*)^r$  such that  $\text{Tors}(V) = \bigcup_j \text{Tors}(Q_j)$ . Consequently, in order to locate the torsion points on an arbitrary intersection of subvarieties,  $V_1 \cap \dots \cap V_k$ , it is enough to find the torsion-translated subtori  $Q_{i,j}$  corresponding to each  $V_i$ , and then taking the torsion points on the variety  $\bigcap_i (\bigcup_j Q_{i,j})$ .

**1.2. Pontrjagin duality.** We start by formalizing the correspondence between the category of abelian complex algebraic reductive groups and the category of finitely generated abelian groups. This well-known correspondence (sometimes called Pontrjagin duality) is based on the functor  $\text{Hom}(-, \mathbb{C}^*)$ .

For a fixed algebraic group  $T := \widehat{H} = \text{Hom}_{\text{group}}(H, \mathbb{C}^*)$ , there is a duality

$$(1) \quad \begin{array}{ccc} & \xrightarrow{\varepsilon} & \\ \text{Algebraic subgroups of } T & & \text{Subgroups of } H \\ & \xleftarrow{V} & \end{array}$$

where  $\varepsilon$  sends  $W \subseteq T$  to  $\ker(\text{Hom}_{\text{alg}}(T, \mathbb{C}^*) \rightarrow \text{Hom}_{\text{alg}}(W, \mathbb{C}^*))$ , while  $V$  sends  $\xi \leq H$  to  $\text{Hom}_{\text{group}}(H/\xi, \mathbb{C}^*)$ . Both sides of (1) are partially ordered sets, with naturally defined meets and joins. As we show in Theorem 2.3, the above correspondence is an order-reversing equivalence of lattices.

For any subgroup  $\xi$  of  $H$ , let  $\bar{\xi} := \{x \in H \mid nx \in \xi \text{ for some } n \in \mathbb{N}\}$ ; the subgroup  $\xi$  is called primitive if  $\bar{\xi} = \xi$ . Under the correspondence  $H \leftrightarrow T$ , primitive subgroups of  $H$  correspond to connected algebraic subgroups of  $T$ . In general, the components of  $V(\xi)$  are indexed by the “determinant group,”  $\bar{\xi}/\xi$ , while the identity component is  $V(\bar{\xi})$ .

**1.3. Intersections of translated subgroups.** Building on the approach taken by E. Hironaka in [3], we use Pontrjagin duality to study intersections of translated subtori in a complex algebraic torus, and, more generally, translated subgroups in an abelian algebraic reductive group over  $\mathbb{C}$ . This allows us to decide whether a finite collection of translated subgroups intersect non-trivially, and, if so, what the dimension of their intersection is.

More precisely, let  $\xi_1, \dots, \xi_k$  be subgroups of  $H$ , and let  $\xi$  be their sum. Let  $\sigma: \xi_1 \times \dots \times \xi_k \rightarrow \xi$  be the sum homomorphism, and let  $\gamma: \xi_1 \times \dots \times \xi_k \rightarrow H^k$  be the product of the inclusion maps. Finally, let  $\eta_1, \dots, \eta_k$  be elements in  $T = \widehat{H}$ , and  $\eta = (\eta_1, \dots, \eta_k) \in T^k$ .

**Theorem A** (Theorem 5.3). The intersection  $Q = \eta_1 V(\xi_1) \cap \dots \cap \eta_k V(\xi_k)$  is non-empty if and only if  $\hat{\gamma}(\eta) \in \text{im}(\hat{\sigma})$ , in which case  $Q = \rho V(\xi)$ , for some  $\rho \in Q$ . Furthermore, if the intersection is non-empty, then

- (1)  $Q$  decomposes into irreducible components as  $Q = \bigcup_{\tau \in \bar{\xi}/\xi} \rho \tau V(\bar{\xi})$ , and  $\dim(Q) = \dim(T) - \text{rank}(\xi)$ .
- (2) If  $\eta$  has finite order, then  $\rho$  can be chosen to have finite order, too. Moreover,  $\text{ord}(\eta) \mid \text{ord}(\rho) \mid c \cdot \text{ord}(\eta)$ , where  $c$  is the largest order of any element in  $\bar{\xi}/\xi$ .

As a corollary, we obtain a general description of the intersection of two arbitrary unions of translated subgroups,  $W = \bigcup_i \eta_i V(\xi_i)$  and  $W' = \bigcup_j \eta'_j V(\xi'_j)$ :

$$(2) \quad W \cap W' = \bigcup_{i,j: \hat{\gamma}_{i,j}(\eta_i, \eta'_j) \in \text{im}(\hat{\sigma}_{i,j})} \eta_{i,j} V(\xi_i + \xi'_j).$$

Moreover, if all the elements  $\eta_i$  and  $\eta'_j$  have finite order, so do the elements  $\eta_{i,j}$ .

In the case when  $H = \mathbb{Z}^r$  and  $T = (\mathbb{C}^*)^r$ , Theorem A recovers a result from [3].

**1.4. Exponential interpretation.** Next, we turn to the relationship between the correspondence  $H \leftrightarrow T$  and the exponential map  $\exp: \text{Lie}(T) \rightarrow T$ .

Setting  $\mathcal{H} = H^\vee := \text{Hom}(H, \mathbb{Z})$ , the exponential map may be identified with the map  $\text{Hom}(\mathcal{H}^\vee, \mathbb{C}) \rightarrow \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*)$  induced by  $\mathbb{C} \rightarrow \mathbb{C}^*$ ,  $z \mapsto e^{2\pi i z}$ . Furthermore, if  $\chi \leq \mathcal{H}$  is a sublattice, then  $V((\mathcal{H}/\chi)^\vee) = \exp(\chi \otimes \mathbb{C})$ .

**Theorem B** (Theorem 6.3). Let  $T$  be a complex abelian reductive group, and let  $\chi_1$  and  $\chi_2$  be two sublattices of  $\mathcal{H} = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)^\vee$ .

(1) Set  $\xi = (\mathcal{H}/\chi_1)^\vee + (\mathcal{H}/\chi_2)^\vee$ . We then have an equality of algebraic groups,

$$\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) = \widehat{\xi/\xi} \cdot V(\xi).$$

(2) Now suppose  $\chi_1$  and  $\chi_2$  are primitive sublattices of  $\mathcal{H}$ , with  $\chi_1 \cap \chi_2 = 0$ . We then have an isomorphism of finite abelian groups,

$$\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) \cong \overline{\chi_1 + \chi_2} / \chi_1 + \chi_2.$$

In the case when  $T = (\mathbb{C}^*)^r$ , part (2) recovers (via a different approach) the main result from [6].

**1.5. Intersections of torsion-translated subtori.** In the case when all the translation factors  $\eta_i$  of the subtori  $V(\xi_i)$  appearing in Theorem A are of finite order, we can say more about the intersection  $\bigcap_{i=1}^k \eta_i V(\xi_i)$ . Since intersections of torsion-translated subtori are again torsion-translated subtori, we may assume  $k = 2$ .

Let  $\xi$  be a primitive lattice in  $\mathbb{Z}^r$ . Given a vector  $\lambda \in \mathbb{Q}^r$ , we say  $\lambda$  virtually belongs to  $\xi$  if  $d \cdot \lambda \in \mathbb{Z}^r$ , where  $d$  is the determinant of the matrix  $[\xi \mid \xi_0]$  obtained by concatenating basis vectors for the sublattices  $\xi$  and  $\xi_0 = \mathbb{Q}\lambda \cap \mathbb{Z}^r$ .

**Theorem C** (Theorem 7.4). Let  $\xi_1$  and  $\xi_2$  be two sublattices in  $\mathbb{Z}^r$ . Set  $\varepsilon = \xi_1 \cap \xi_2$ , and write  $\widehat{\varepsilon/\varepsilon} = \{\exp(2\pi i \mu_k)\}_{k=1}^s$ . Also let  $\eta_1$  and  $\eta_2$  be two torsion elements in  $(\mathbb{C}^*)^r$ , and write  $\eta_j = \exp(2\pi i \lambda_j)$ . The following are equivalent:

- (1) The variety  $Q = \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2)$  is non-empty.
- (2) One of the vectors  $\lambda_1 - \lambda_2 - \mu_k$  virtually belongs to the lattice  $(\mathbb{Z}^r/\varepsilon)^\vee$ .

If either condition is satisfied, then  $Q = \rho V(\xi_1 + \xi_2)$  for some  $\rho \in Q$ .

This theorem provides an algorithm for checking the condition from Theorem A, solely in terms of arithmetic data extracted from the lattices  $\xi_1$  and  $\xi_2$  and the rational vectors  $\lambda_1$  and  $\lambda_2$ . The complexity of this algorithm is linear with respect to the order of the determinant group  $\widehat{\varepsilon/\varepsilon}$ .

**1.6. Some applications.** As mentioned in §1.1, one of our main motivations comes from the study of characteristic varieties, especially as it regards the intersection pattern of the components of such varieties, and the count of their torsion points. Theorem A yields several corollaries, which provide very specific information in this direction; this information has since been put to use in [10, 11].

In addition to these applications, we also consider the following counting problem: how many translated subtori does an algebraic torus have, once we fix the identity component, and the number of irreducible components? Using the correspondence from (1), and a classical result on the number of finite-index subgroups of a free abelian group, we express the generating function for this counting problem in terms of the Riemann zeta function.

## 2. FINITELY GENERATED ABELIAN GROUPS AND ABELIAN REDUCTIVE GROUPS

In this section, we describe an order-reversing isomorphism between the lattice of subgroups of a finitely generated abelian group  $H$  and the lattice of algebraic subgroups of the corresponding abelian, reductive, complex algebraic group  $T$ .

**2.1. Abelian reductive groups.** We start by recalling a well-known equivalence between two categories: that of finitely generated abelian groups,  $\text{AbFgGp}$ , and that of abelian, reductive, complex algebraic groups,  $\text{AbRed}$ . For a somewhat similar approach, see also [3] and [5].

Let  $\mathbb{C}^*$  be the multiplicative group of units in the field  $\mathbb{C}$  of complex numbers. Given a group  $G$ , let  $\widehat{G} = \text{Hom}(G, \mathbb{C}^*)$  be the group of complex-valued characters of  $G$ , with pointwise multiplication inherited from  $\mathbb{C}^*$ , and identity the character taking constant value  $1 \in \mathbb{C}^*$  for all  $g \in G$ . If the group  $G$  is finitely generated, then  $\widehat{G}$  is an abelian, complex reductive algebraic group.

Note that  $\widehat{G} \cong \widehat{H}$ , where  $H$  is the maximal abelian quotient of  $G$ . If  $H$  is torsion-free, say,  $H = \mathbb{Z}^r$ , then  $\widehat{H}$  can be identified with the complex algebraic torus  $(\mathbb{C}^*)^r$ . If  $A$  is a finite abelian group, then  $\widehat{A}$  is, in fact, isomorphic to  $A$ .

Given a homomorphism  $\phi: G_1 \rightarrow G_2$ , let  $\hat{\phi}: \widehat{G}_2 \rightarrow \widehat{G}_1$  be the induced morphism between character groups, given by  $\hat{\phi}(\rho) = \rho \circ \phi$ . Since  $\mathbb{C}^*$  is a divisible abelian group, the functor  $H \mapsto \widehat{H} = \text{Hom}(H, \mathbb{C}^*)$  is exact.

Now let  $T$  be an abelian, complex algebraic reductive group. We can then associate to  $T$  its weight group,  $\check{T} = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)$ , where the hom set is taken in the category of algebraic groups. It turns out that  $\check{T}$  is a finitely generated abelian group, which can be described concretely, as follows.

According to the classification of abelian reductive groups over  $\mathbb{C}$  (cf. [9]), the identity component of  $T$  is an algebraic torus, i.e., it is of the form  $(\mathbb{C}^*)^r$  for some integer  $r \geq 0$ . Furthermore, this identity component has to be a normal subgroup. Thus, the algebraic group  $T$  is isomorphic to  $(\mathbb{C}^*)^r \times A$ , for some finite abelian group  $A$ .

The coordinate ring  $O[T]$  decomposes as

$$(3) \quad O[(\mathbb{C}^*)^r \times A] \cong O[(\mathbb{C}^*)^r] \otimes O[A] \cong \mathbb{C}[\mathbb{Z}^r] \otimes \mathbb{C}[\widehat{A}],$$

where  $\mathbb{C}[G]$  denotes the group ring of a group  $G$ . Let  $O[T]^*$  be the group of units in the coordinate ring of  $T$ . By (3), this group is isomorphic to  $\mathbb{C}^* \times \mathbb{Z}^r \times A$ , where  $\mathbb{C}^*$  corresponds to the non-zero constant functions. Taking the quotient by this  $\mathbb{C}^*$  factor, we get isomorphisms

$$(4) \quad \check{T} \cong O[T]^* / \mathbb{C}^* \cong \mathbb{Z}^r \times A.$$

Clearly,  $\text{maxSpec}(\mathbb{C}[\check{T}]) = \text{Hom}_{\text{alg}}(\mathbb{C}[\check{T}], \mathbb{C}) = \text{Hom}_{\text{group}}(\check{T}, \mathbb{C}^*) = T$ .

Now let  $f: T_1 \rightarrow T_2$  be a morphism in  $\text{AbRed}$ . Then the induced morphism on coordinate rings,  $f^*: \mathcal{O}[T_2] \rightarrow \mathcal{O}[T_1]$ , restricts to a group homomorphism,  $f^*: \mathcal{O}[T_2]^* \rightarrow \mathcal{O}[T_1]^*$ , which takes constants to constants. Under the identification from (4),  $f^*$  induces a homomorphism  $\check{f}: \check{T}_2 \rightarrow \check{T}_1$  between the corresponding weight groups.

The following proposition is now easy to check.

**Proposition 2.1.** *The functors  $H \rightsquigarrow \widehat{H}$  and  $T \rightsquigarrow \check{T}$  establish a contravariant equivalence between the category of finitely generated abelian groups and the category of abelian reductive groups over  $\mathbb{C}$ .*

Recall now that the functor  $H \rightsquigarrow \widehat{H}$  is exact. Hence, the functor  $T \rightsquigarrow \check{T}$  is also exact.

**Remark 2.2.** The above functors behave well with respect to (finite) direct products. For instance, let  $\alpha: A \rightarrow C$  and  $\beta: B \rightarrow C$  be two homomorphisms between finitely generated abelian groups, and consider the homomorphism  $\delta: A \times B \rightarrow C$  defined by  $\delta(a, b) = \alpha(a) + \beta(b)$ . The morphism  $\widehat{\delta}: \widehat{A \times B} = \widehat{A} \times \widehat{B} \rightarrow \widehat{C}$  is then given by  $\widehat{\delta}(f) = (\widehat{\alpha}(f), \widehat{\beta}(f))$ .

**2.2. The lattice of subgroups of a finitely generated abelian group.** Recall that a poset  $(L, \leq)$  is a *lattice* if every pair of elements has a least upper bound and a greatest lower bound. Define operations  $\vee$  and  $\wedge$  on  $L$  (called join and meet, respectively) by  $x \vee y = \sup\{x, y\}$  and  $x \wedge y = \inf\{x, y\}$ .

The lattice  $L$  is called *modular* if, whenever  $x < z$ , then  $x \vee (y \wedge z) = (x \vee y) \wedge z$ , for all  $y \in L$ . The lattice  $L$  is called *distributive* if  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$  and  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ , for all  $x, y, z \in L$ . A lattice is modular if and only if it does not contain the pentagon as a sublattice, whereas a modular lattice is distributive if and only if it does not contain the diamond as a sublattice.

Finally,  $L$  is said to be a *ranked lattice* if there is a function  $r: L \rightarrow \mathbb{Z}$  such that  $r$  is constant on all minimal elements,  $r$  is monotone (if  $\xi_1 \leq \xi_2$ , then  $r(\xi_1) \leq r(\xi_2)$ ), and  $r$  preserves covering relations (if  $\xi_1 \leq \xi_2$ , but there is no  $\xi$  such that  $\xi_1 < \xi < \xi_2$ , then  $r(\xi_2) = r(\xi_1) + 1$ ).

Given a group  $G$ , the set of subgroups of  $G$  forms a lattice,  $\mathcal{L}(G)$ , with order relation given by inclusion. The join of two subgroups,  $\gamma_1$  and  $\gamma_2$ , is the subgroup generated by  $\gamma_1$  and  $\gamma_2$ , and their meet is the intersection  $\gamma_1 \cap \gamma_2$ . There is unique minimal element—the trivial subgroup, and a unique maximal element—the group  $G$  itself. The lattice  $\mathcal{L}(G)$  is distributive if and only if  $G$  is locally cyclic (i.e., every finitely generated subgroup is cyclic). Similarly, one may define the lattice of normal subgroups of  $G$ ; this lattice is always modular. We refer to [8] for more on all this.

Now let  $H$  be a finitely generated abelian group, and let  $\mathcal{L}(H)$  be its lattice of subgroups. In this case, the join of two subgroup,  $\xi_1$  and  $\xi_2$ , equals the sum  $\xi_1 + \xi_2$ . By the above, the lattice  $\mathcal{L}(H)$  is always modular, but it is not a distributive lattice, unless  $H$  is cyclic. Furthermore,  $\mathcal{L}(H)$  is a ranked lattice, with rank function  $\xi \mapsto \text{rank}(\xi) = \dim_{\mathbb{Q}}(\xi \otimes \mathbb{Q})$  enjoying the following property:  $\text{rank}(\xi_1) + \text{rank}(\xi_2) = \text{rank}(\xi_1 \wedge \xi_2) + \text{rank}(\xi_1 \vee \xi_2)$ .

**2.3. The lattice of algebraic subgroups of a complex algebraic torus.** Now let  $T$  be a complex abelian reductive group, and let  $(\mathcal{L}_{\text{alg}}(T), \leq)$  be the poset of algebraic subgroups of  $T$ , ordered by inclusion. It is readily seen that  $\mathcal{L}_{\text{alg}}(T)$  is a ranked modular lattice.

For any algebraic subgroups  $P_1$  and  $P_2$  of  $T$ , the join  $P_1 \vee P_2 = P_1 \cdot P_2$  is the algebraic subgroup generated by the two subgroups  $P_1$  and  $P_2$ , while the meet  $P_1 \wedge P_2 = P_1 \cap P_2$  is the intersection of the two subgroups. Furthermore, the rank of a subgroup  $P$  is its dimension.

The next theorem shows that the natural correspondence from Proposition 2.1 is lattice-preserving. Recall that, if  $H$  is a finitely generated abelian group, the character group  $\widehat{H} = \text{Hom}_{\text{group}}(H, \mathbb{C}^*)$  is an abelian reductive group over  $\mathbb{C}$ , and conversely, if  $T$  is an abelian reductive group, the weight group  $\check{T} = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)$  is a finitely generated abelian group.

**Theorem 2.3.** *Suppose  $H \cong \check{T}$ , or equivalently,  $T \cong \widehat{H}$ . There is then an order-reversing isomorphism between the lattice of subgroups of  $H$  and the lattice of algebraic subgroups of  $T$ .*

*Proof.* For any subgroup  $\xi \leq H$ , let

$$(5) \quad V(\xi) = \max\text{Spec}(\mathbb{C}[H/\xi])$$

be the set of closed points of  $\text{Spec}(\mathbb{C}[H/\xi])$ . Clearly, the variety  $V(\xi)$  embeds into  $\max\text{Spec}(\mathbb{C}[H]) \cong T$  as an algebraic subgroup. This subgroup can be naturally identified with the group of characters of  $H/\xi$ , that is,  $V(\xi) \cong \widehat{H/\xi}$ , or equivalently,

$$(6) \quad \widehat{\xi} \cong T/V(\xi).$$

For any algebraic subset  $W \subset T$ , define

$$(7) \quad \epsilon(W) = \ker(\text{Hom}_{\text{alg}}(T, \mathbb{C}^*) \rightarrow \text{Hom}_{\text{alg}}(W, \mathbb{C}^*)).$$

Write  $T = (\mathbb{C}^*)^r \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_s}$ , where  $\mathbb{Z}_{k_i}$  embeds in  $\mathbb{C}^*$  as the subgroup of  $k_i$ -th roots of unity. Using the standard coordinates of  $(\mathbb{C}^*)^{r+s}$ , we can identify  $\epsilon(W)$  with the subgroup  $\{\lambda \in H : t^\lambda - 1 \text{ vanishes on } W\}$ .

The proof of the theorem is completed by the next three lemmas.  $\square$

**Lemma 2.4.** *If  $\xi$  is a subgroup of  $H$ , then  $\epsilon(V(\xi)) = \xi$ .*

*Proof.* The inclusion  $\epsilon(V(\xi)) \supseteq \xi$  is clear. Now suppose  $\lambda \in H \setminus \xi$ . Then we may define a character  $\rho \in \widehat{H}$  such that  $\rho(\lambda) \neq 1$ , but  $\rho(\xi) = 1$ . Evidently,  $\lambda \notin \epsilon(V(\xi))$ , and we are done.  $\square$

Given two algebraic subgroups,  $P$  and  $Q$  of  $T$ , let  $\text{Hom}_{\text{alg}}(P, Q)$  be the set of morphisms from  $P$  to  $Q$  which preserve both the algebraic and multiplicative structure.

**Lemma 2.5.** *Any algebraic subgroup  $P$  of  $T$  is of the form  $V(\xi)$ , for some subgroup  $\xi \subseteq H$ .*

*Proof.* Let  $\iota: P \hookrightarrow T$  be the inclusion map. Applying the functor  $\text{Hom}_{\text{alg}}(-, \mathbb{C}^*)$ , we obtain an epimorphism  $\iota^*: H \twoheadrightarrow \text{Hom}_{\text{alg}}(P, \mathbb{C}^*)$ . Set  $\xi = \ker(\iota^*)$ ; then  $P = V(\xi)$ .  $\square$

The above two lemmas (which generalize Lemmas 3.1 and 3.2 from [3]), show that we have a natural correspondence between algebraic subgroups of  $T$  and subgroups of  $H$ . This correspondence preserves the lattice structure on both sides. That is, if  $\xi_1 \leq \xi_2$ , then  $V(\xi_1) \supseteq V(\xi_2)$ , and similarly, if  $P_1 \subseteq P_2$ , then  $\epsilon(P_1) \supseteq \epsilon(P_2)$ .



**Lemma 2.6.** *The natural correspondence between algebraic subgroups of  $T$  and subgroups of  $H$  is an order-reversing lattice isomorphism. In particular,*

$$V(\xi_1 + \xi_2) = V(\xi_1) \cap V(\xi_2) \quad \text{and} \quad V(\xi_1 \cap \xi_2) = V(\xi_1) \cdot V(\xi_2).$$

*Proof.* The first claim follows from the two lemmas above. The two equalities are consequences of this.  $\square$

The above correspondence reverses ranks, i.e.,  $\text{rank}(\xi) = \text{codim } V(\xi)$  and  $\text{corank}(\xi) = \dim V(\xi)$ .

**2.4. Counting algebraic subtori.** As a quick application, we obtain a counting formula for the number of algebraic subgroups of an  $r$ -dimensional complex algebraic torus, having precisely  $k$  connected components, and a fixed,  $n$ -dimensional subtorus as the identity component. It is convenient to restate such a problem in terms of a zeta function.

**Definition 2.7.** Let  $T$  be an abelian reductive group, and let  $T_0$  be a fixed connected algebraic subgroup. Define the zeta function of this pair as

$$\zeta(T, T_0, s) = \sum_{k=1}^{\infty} \frac{a_k(T, T_0)}{k^s},$$

where  $a_k(T, T_0)$  is the number of algebraic subgroups  $W \leq T$  with identity component equal to  $T_0$  and such that  $|W/T_0| = k$ .

This definition is modeled on that of the zeta function of a finitely generated group  $G$ , which is given by  $\zeta(G, s) = \sum_{k=1}^{\infty} a_k(G)k^{-s}$ , where  $a_k(G)$  is the number of index- $k$  subgroups of  $G$ , see for instance [7].

**Corollary 2.8.** *Suppose  $T \cong (\mathbb{C}^*)^r$  and  $T_0 \cong (\mathbb{C}^*)^n$ , for some  $0 \leq n \leq r$ . Then  $\zeta(T, T_0, s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-r+n+1)$ , where  $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$  is the usual Riemann zeta function.*

*Proof.* First assume  $n = 0$ , so that  $T_0 = \{1\}$ . By Theorem 4.3,  $a_k(T, \{1\})$  is the number of algebraic subgroups  $W \leq T$  of the form  $W = V(\xi)$ , where  $\xi \leq \mathbb{Z}^r$  and  $[\mathbb{Z}^r : \xi] = k$ . Clearly, this number equals  $a_k(\mathbb{Z}^r)$ , and so  $\zeta(T, \{1\}, s) = \zeta(\mathbb{Z}^r, s)$ . By a classical result of Bushnell and Reiner (see [7]), we have that  $\zeta(\mathbb{Z}^r, s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-r+1)$ . This proves the claim for  $n = 0$ .

For  $n > 0$ , we simply take the quotient of the group  $T$  by the fixed subtorus  $T_0$ , to get  $\zeta(T, T_0, s) = \zeta(T/T_0, \{1\}, s) = \zeta(\mathbb{Z}^{r-n}, s)$ . This ends the proof.  $\square$

### 3. PRIMITIVE LATTICES AND CONNECTED SUBGROUPS

In this section, we show that the correspondence between  $\mathcal{L}(H)$  and  $\mathcal{L}_{\text{alg}}(T)$  restricts to a correspondence between the primitive subgroups of  $H$  and the connected algebraic subgroups of  $T$ . We also explore the relationship between the connected components of  $V(\xi)$  and the determinant group,  $\bar{\xi}/\xi$ , of a subgroup  $\xi \leq H$ .

**3.1. Primitive subgroups.** As before, let  $H$  be a finitely generated abelian group. We say that a subgroup  $\xi \leq H$  is *primitive* if there is no other subgroup  $\xi' \leq H$  with  $\xi < \xi'$  and  $[\xi' : \xi] < \infty$ . In particular, a primitive subgroup must contain the torsion subgroup,  $\text{Tors}(H) = \{\lambda \in H \mid \exists n \in \mathbb{N} \text{ such that } n\lambda = 0\}$ .

The intersection of two primitive subgroups is again a primitive subgroup, but the sum of two primitive subgroups need not be primitive (see, for instance, the proof of Corollary 6.6). Thus, the set of primitive subgroups of  $H$  is not necessarily a sublattice of  $\mathcal{L}(H)$ .

When  $H$  is free abelian, then all subgroups  $\xi \leq H$  are also free abelian. In this case,  $\xi$  is primitive if and only if it has a basis that can be extended to a basis of  $H$ , or, equivalently,  $H/\xi$  is torsion-free. It is customary to call such a subgroup a *primitive lattice*.

Returning to the general situation, let  $H$  be a finitely generated abelian group. Given an arbitrary subgroup  $\xi \leq H$ , define its primitive closure,  $\bar{\xi}$ , to be the smallest primitive subgroup of  $H$  containing  $\xi$ . Clearly,

$$(8) \quad \bar{\xi} = \{\lambda \in H : \exists n \in \mathbb{N} \text{ such that } n\lambda \in \xi\}.$$

Note that  $H/\bar{\xi}$  is torsion-free, and thus we have a split exact sequence,

$$(9) \quad 0 \longrightarrow \bar{\xi} \longrightarrow H \xrightarrow{\quad \text{pr} \quad} H/\bar{\xi} \longrightarrow 0.$$

By definition,  $\xi$  is a finite-index subgroup of  $\bar{\xi}$ ; in particular,  $\text{rank}(\xi) = \text{rank}(\bar{\xi})$ . We call the quotient group,  $\bar{\xi}/\xi$ , the *determinant group* of  $\xi$ . We have an exact sequence,

$$(10) \quad 0 \longrightarrow H/\bar{\xi} \longrightarrow H/\xi \longrightarrow \bar{\xi}/\xi \longrightarrow 0,$$

with  $\bar{\xi}/\xi$  finite. The inclusion  $\bar{\xi} \hookrightarrow H$  induces a monomorphism  $\bar{\xi}/\xi \hookrightarrow H/\xi$ , which yields a splitting for the above sequence, showing that  $\bar{\xi}/\xi \cong \text{Tors}(H/\xi)$ . Since the group  $\bar{\xi}/\xi$  is finite, it is isomorphic to its character group,  $\widehat{\bar{\xi}/\xi}$ , which in turn can be viewed as a (finite) subgroup of  $\widehat{H} = T$ .

Using an approach similar to the one from [3, Lemma 3.3], we sharpen and generalize that result, as follows.

**Lemma 3.1.** *For every subgroup  $\xi \leq H$ , we have an isomorphism of algebraic groups,*

$$(11) \quad V(\xi) \cong \widehat{\bar{\xi}/\xi} \cdot V(\bar{\xi}).$$

Moreover,

- (1)  $V(\xi) = \bigcup_{\rho \in \widehat{\bar{\xi}/\xi}} \rho V(\bar{\xi})$  is the decomposition of  $V(\xi)$  into irreducible components, with  $V(\bar{\xi})$  as the component of the identity.
- (2)  $V(\xi)/V(\bar{\xi}) \cong \widehat{\bar{\xi}/\xi}$ . In particular, if  $\text{rank } \xi = \dim H$ , then  $V(\xi) \cong \widehat{\bar{\xi}/\xi}$ .

*Proof.* As noted in §2.1, the functor  $\text{Hom}(-, \mathbb{C}^*)$  is exact. Applying this functor to sequence (10), we obtain an exact sequence in  $\text{AbRed}$ ,

$$(12) \quad 0 \longrightarrow \widehat{\bar{\xi}/\xi} \longrightarrow \widehat{H/\xi} \longrightarrow \widehat{H/\bar{\xi}} \longrightarrow 0.$$

$\parallel$   
 $V(\xi)$

$\parallel$   
 $V(\bar{\xi})$



Since sequence (10) is split, sequence (12) is also split, and thus we get decomposition (11).

Now recall that  $H/\bar{\xi}$  is torsion-free; thus,  $V(\bar{\xi}) = \max\text{Spec}(\mathbb{C}[H/\bar{\xi}])$  is a connected algebraic subgroup of  $T$ . Claims (1) and (2) readily follow.  $\square$

**Example 3.2.** Let  $H = \mathbb{Z}$  and identify  $\widehat{H} = \mathbb{C}^*$ . If  $\xi = 2\mathbb{Z}$ , then  $V(\xi) = \{\pm 1\} \subset \mathbb{C}^*$ , whereas  $\bar{\xi} = H$  and  $V(\bar{\xi}) = \{1\} \subset \mathbb{C}^*$ .

Clearly, the subgroup  $\xi$  is primitive if and only if  $\xi = \bar{\xi}$ . Thus,  $\xi$  is primitive if and only if the variety  $V(\xi)$  is connected. Putting things together, we obtain the following corollary to Theorem 2.3 and Lemma 3.1.

**Corollary 3.3.** *Let  $H$  be a finitely generated abelian group, and let  $T = \widehat{H}$ . The natural correspondence between  $\mathcal{L}(H)$  and  $\mathcal{L}_{\text{alg}}(T)$  restricts to a correspondence between the primitive subgroups of  $H$  and the connected algebraic subgroups of  $T$ .*

**3.2. The dual lattice.** Given an abelian group  $A$ , let  $A^\vee = \text{Hom}(A, \mathbb{Z})$  be the dual group. Clearly, if  $H$  is a finitely generated abelian group, then  $H^\vee$  is torsion-free, with  $\text{rank } H^\vee = \text{rank } H$ .

Now suppose  $\xi \leq H$  is a subgroup. By passing to duals, the projection map  $\pi: H \rightarrow H/\xi$  yields a monomorphism  $\pi^\vee: (H/\xi)^\vee \hookrightarrow H^\vee$ . Thus,  $(H/\xi)^\vee$  can be viewed in a natural way as a subgroup of  $H^\vee$ . In fact, more is true.

**Lemma 3.4.** *Let  $H$  be a finitely generated abelian group, and let  $\xi \leq H$  be a subgroup. Then  $(H/\xi)^\vee$  is a primitive lattice in  $H^\vee$ .*

*Proof.* Dualizing the short exact sequence  $0 \rightarrow \xi \rightarrow H \xrightarrow{\pi} H/\xi \rightarrow 0$ , we obtain a long exact sequence,

$$(13) \quad 0 \longrightarrow (H/\xi)^\vee \xrightarrow{\pi^\vee} H^\vee \longrightarrow \xi^\vee \longrightarrow \text{Ext}(H/\xi, \mathbb{Z}) \longrightarrow 0.$$

Upon identifying  $\text{Ext}(H/\xi, \mathbb{Z}) = \text{Tors}(H/\xi) = \bar{\xi}/\xi$  and setting  $K = \text{coker}(\pi^\vee)$ , the above sequence splits into two short exact sequences,  $0 \rightarrow (H/\xi)^\vee \rightarrow H^\vee \rightarrow K \rightarrow 0$  and

$$(14) \quad 0 \longrightarrow K \longrightarrow \xi^\vee \longrightarrow \bar{\xi}/\xi \longrightarrow 0.$$

Now, since  $K$  is a subgroup of  $\xi^\vee$ , it must be torsion free. Thus,  $(H/\xi)^\vee$  is a primitive lattice in  $H^\vee$ .  $\square$

Given two primitive subgroups  $\xi_1, \xi_2 \leq H$ , their sum,  $\xi_1 + \xi_2$ , may not be a primitive subgroup of  $H$ . Likewise, although both  $(H/\xi_1)^\vee$  and  $(H/\xi_2)^\vee$  are primitive subgroups of  $H^\vee$ , their sum may not be primitive. Nevertheless, the following lemma shows that the respective determinant groups are the same.

**Proposition 3.5.** *Let  $H$  be a finitely generated abelian group, and let  $\xi_1$  and  $\xi_2$  be primitive subgroups of  $H$ , with  $\xi_1 \cap \xi_2$  finite. Set  $\xi = (H/\xi_1)^\vee + (H/\xi_2)^\vee$ , and let  $\bar{\xi}$  be the primitive closure of  $\xi$  in  $H^\vee$ . Then*

$$\bar{\xi}/\xi \cong \overline{\xi_1 + \xi_2}/\xi_1 + \xi_2.$$

*Proof.* Replacing  $H$  by  $H/\text{Tors}(H)$  and  $\xi_i$  by  $\xi_i/\text{Tors}(H)$  if necessary, we may assume that  $H$  is free abelian, and  $\xi_1$  and  $\xi_2$  are primitive lattices with  $\xi_1 \cap \xi_2 = \{0\}$ . Furthermore, choosing a splitting of  $H^\vee/\bar{\xi} \hookrightarrow H^\vee$  if necessary, we may assume that  $\bar{\xi} \cong H^\vee$ , or equivalently,  $\bar{\xi}_1 + \bar{\xi}_2 = H$ . Write  $s = \text{rank } \xi_1$  and  $t = \text{rank } \xi_2$ . Then  $s + t = n$ , where  $n = \text{rank } H$ .

Choose a basis  $\{e_1, \dots, e_s\}$  for  $\xi_1$ . Since  $\xi_1$  is a primitive lattice in  $H$ , we may extend this basis to a basis  $\{e_1, \dots, e_s, f_1, \dots, f_t\}$  for  $H$ . Picking a suitable basis for  $\xi_2$ , we may assume that the inclusion  $\iota: \xi_1 + \xi_2 \hookrightarrow H$  is given by a matrix of the form

$$(15) \quad \left( \begin{array}{c|c} I_s & C \\ \hline 0 & D \end{array} \right),$$

where  $I_s$  is the  $s \times s$  identity matrix and  $D = \text{diag}(d_1, \dots, d_t)$  is a diagonal matrix with positive diagonal entries  $d_1, \dots, d_t$  such that  $1 = d_1 = \dots = d_{m-1}$  and  $1 \neq d_t \mid d_{t-1} \mid \dots \mid d_m$ , for some  $1 \leq m \leq t+1$ . Then  $\bar{\xi}_1 + \bar{\xi}_2/\bar{\xi}_1 + \bar{\xi}_2 = \mathbb{Z}/d_m\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_t\mathbb{Z}$ .

Notice that the columns of the matrix  $\begin{pmatrix} C \\ D \end{pmatrix}$  form a basis for  $\xi_2$ . Since  $\xi_2$  is a primitive lattice in  $H$ , the last  $t-m$  columns of  $C$  must have a minor of size  $t-m$  equal to  $\pm 1$ . Without loss of generality, we may assume that the corresponding rows are also the last  $t-m$  ones.

The canonical projection  $\pi: H \twoheadrightarrow H/\xi_1 + H/\xi_2$  is given by a matrix of the form

$$(16) \quad \left( \begin{array}{c|c} X & Y \\ \hline 0 & I_t \end{array} \right).$$

Using row and column operations, the matrix  $X$  can be brought to the diagonal form  $\text{diag}(x_1, \dots, x_s)$ , where  $x_i$  are positive integers with  $1 = x_1 = \dots = x_{a-1}$  and  $1 \neq x_s \mid x_{s-1} \mid \dots \mid x_a$ . Moreover, the submatrix of  $Y$  involving the last  $s-a+1$  rows and columns is invertible. Taking the dual basis of  $H^\vee = \text{Hom}(H, \mathbb{Z})$ , the inclusion  $\xi = (H/\xi_1)^\vee + (H/\xi_2)^\vee \hookrightarrow H^\vee$  is given by the matrix  $\begin{pmatrix} X^T & 0 \\ Y^T & I_t \end{pmatrix}$ . It follows that  $\bar{\xi}/\xi = \mathbb{Z}/x_a\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/x_s\mathbb{Z}$ .

Evidently, the restriction of  $\pi \circ \iota$  to  $\xi_2$  is the zero map; hence,  $XC = -YD$ . For a fixed integer  $k \leq s$ , set  $\delta_k := d_t d_{t-1} \dots d_{k-s+t}$  and  $y_k := x_s x_{s-1} \dots x_k$ . Clearly,  $\delta_k$  is the gcd of all minors of size  $s-k+1$  of the submatrix in  $YD$  involving the last  $s-a+1$  rows and columns. Hence,  $\delta_k$  is also the gcd of all minors of size  $s-k+1$  of the corresponding submatrix of  $XC$ . Thus,  $\delta_k = y_k$ . Hence,  $d_t = x_s, \dots, d_m = x_{s-t+m}$ , and  $1 = d_{m-1} = x_{s-t+m-1}$ , which implies  $s-t+m = a$ . This yields the desired conclusion.  $\square$

#### 4. CATEGORICAL REFORMULATION

In this section, we reformulate Theorem 2.3 using the language of categories. In order to simultaneously consider the category of all finitely generated abelian groups, and the lattice structure for all the subgroups of a fixed abelian group, we need the language of fibered categories from [2, §5.1], which we briefly recall here.

**4.1. Fibered categories.** Recall that a poset  $(L, \leq)$  can be seen as a small category, with objects the same as the elements of  $L$ , and with an arrow  $p \rightarrow p'$  in the category  $L$  if and only if  $p \leq p'$ .

Let  $\mathcal{E}$  be a category. We denote by  $\text{Ob } \mathcal{E}$  the objects of  $\mathcal{E}$ , and by  $\text{Mor } \mathcal{E}$  its morphisms. A *category over  $\mathcal{E}$*  is a category  $\mathcal{F}$ , together with a functor  $\Phi: \mathcal{F} \rightarrow \mathcal{E}$ . For  $T \in \text{Ob } \mathcal{E}$ , we denote by  $\mathcal{F}(T)$  the subcategory of  $\mathcal{F}$  consisting of objects  $\xi$  with  $\Phi(\xi) = T$ , and morphisms  $f$  with  $\Phi(f) = \text{id}_T$ .

**Definition 4.1.** Let  $\Phi: \mathcal{F} \rightarrow \mathcal{E}$  be a category over  $\mathcal{E}$ . Let  $f: v \rightarrow u$  be a morphism in  $\mathcal{F}$ , and set  $(F: V \rightarrow U) = (\Phi(f: v \rightarrow u))$ . Then  $f$  is said to be a *Cartesian morphism* if for any  $h: v' \rightarrow u$  with  $\Phi(h: v' \rightarrow u) = F$ , there exists a unique  $h': v' \rightarrow v$  in  $\text{Mor } \mathcal{F}(F)$  such that  $h = f \circ h'$ .

The above definition is summarized in the following diagram:

$$(17) \quad \begin{array}{ccc} v' & & \\ \downarrow & \searrow h & \\ h' \downarrow & & u \\ v & \xrightarrow{f} & u \\ & \xrightarrow{F} & U \end{array}$$

**Definition 4.2.** We say that  $\Phi: \mathcal{F} \rightarrow \mathcal{E}$  is a *fibred category* if for any morphism  $F: V \rightarrow U$  in  $\mathcal{E}$ , and any object  $u \in \text{Ob}(\mathcal{F}(U))$ , there exists a Cartesian morphism  $f: v \rightarrow u$ , with  $\Phi(f) = F$ . Moreover, the composition of Cartesian morphisms is required to be a Cartesian morphism.

We say  $\mathcal{F}$  is a *lattice over  $\mathcal{E}$*  (or less succinctly, a category fibered in lattices over  $\mathcal{E}$ ) if  $\mathcal{F}$  is a fibred category, and every  $\mathcal{F}(T)$  is a lattice.

**4.2. A lattice over  $\text{AbRed}$ .** We now construct a category fibered in lattices over the category of abelian complex algebraic reductive groups,  $\text{AbRed}$ . Let  $\text{SubAbRed}$  be the category with objects

$$(18) \quad \text{Ob } \text{SubAbRed} = \{i: W \hookrightarrow P \mid i \text{ is a closed immersion of algebraic subgroups}\},$$

and morphisms between  $i: W \rightarrow P$  and  $i': W' \rightarrow P'$  the set of pairs

$$(19) \quad \{(f, g) \mid f: W \rightarrow W' \text{ and } g: P \rightarrow P' \text{ such that } i' \circ f = g \circ i\}.$$

Projection to the target,

$$(20) \quad \begin{array}{ccc} W \hookrightarrow P & \mapsto & P \\ \downarrow f & & \downarrow g \\ W' \hookrightarrow P' & & P' \end{array}$$

defines a functor from  $\text{SubAbRed}$  to  $\text{AbRed}$ . One can easily check that this functor is a category fibered in lattices over  $\text{AbRed}$ , with Cartesian morphisms obtained by taking preimages of subtori. More precisely, suppose  $F: W \rightarrow P$  is a morphism in  $\text{AbRed}$ , and

$\theta \hookrightarrow P$  is an object in  $\text{Ob SubAbRed}$ ; the corresponding Cartesian morphism is then

$$(21) \quad \begin{array}{ccc} F^{-1}(\theta) & \xrightarrow{F} & \theta \\ \downarrow i & & \downarrow i \\ W & \xrightarrow{F} & P \end{array}$$

A similar construction works for  $\text{AbFgGp}$ , the category of finitely generated abelian groups. That is, we can construct a category  $\text{SubAbGp}$  fibered in lattices over  $\text{AbFgGp}$  by taking injective morphisms of finitely generated abelian groups.

**4.3. An equivalence of fibered categories.** We now construct an explicit (contravariant) equivalence between these two fibered categories considered above. For any inclusion of subgroups  $\eta \hookrightarrow \xi$ , let

$$(22) \quad V(\eta \hookrightarrow \xi) := (\max\text{Spec}(\mathbb{C}[\xi/\eta]) \hookrightarrow \max\text{Spec}(\mathbb{C}[\xi])).$$

This algebraic subgroup is naturally identified with the subgroup of characters of  $\xi/\eta$ , i.e., the closed embedding of subtori  $\text{Hom}(\xi/\eta, \mathbb{C}^*) \rightarrow \text{Hom}(\xi, \mathbb{C}^*)$ .

Finally, for any algebraic subgroups  $W \hookrightarrow P$ , let

$$(23) \quad \epsilon(W \hookrightarrow P) := (\{\lambda \in \text{Hom}(P, \mathbb{C}^*) : \lambda^l - 1 \text{ vanishes on } W\} \subseteq \text{Hom}(P, \mathbb{C}^*)).$$

For a fixed algebraic group  $T$  in  $\text{AbRed}$ , with character group  $H = \check{T}$ , the subgroup  $\epsilon(W \hookrightarrow T)$  of  $H$  coincides with the subgroup  $\epsilon(W)$  defined in (7), and the algebraic subgroup  $V(\xi \hookrightarrow H)$  of  $T$  coincides with the algebraic subgroup  $V(\xi)$  defined in (5).

Tracing through the definitions, we obtain the following result, which reformulates Theorem 2.3 in this setting.

**Theorem 4.3.** *The two fibered categories  $\text{SubAbRed}$  and  $\text{SubAbGp}$  are equivalent, with (contravariant) equivalences given by the functors  $V$  and  $\epsilon$  defined above.*

## 5. INTERSECTIONS OF TRANSLATED ALGEBRAIC SUBGROUPS

In Sections 2 and 3, we only considered intersections of algebraic subgroups. In this section, we consider the more general situation where translated subgroups intersect.

**5.1. Two morphisms.** As usual, let  $T$  be a complex abelian reductive group, and let  $H = \check{T}$  be the weight group corresponding to  $T = \widehat{H}$ .

Let  $\xi_1, \dots, \xi_k$  be subgroups of  $H$ . Set  $\xi = \xi_1 + \dots + \xi_k$ , and let  $\sigma: \xi_1 \times \dots \times \xi_k \rightarrow \xi$  be the homomorphism given by  $(\lambda_1, \dots, \lambda_k) \mapsto \lambda_1 + \dots + \lambda_k$ . Consider the induced morphism on character groups,

$$(24) \quad \hat{\sigma}: \widehat{\xi} \longrightarrow \widehat{\xi_1} \times \dots \times \widehat{\xi_k}.$$

Using Remark 2.2, the next lemma is readily verified.

**Lemma 5.1.** *Under the isomorphisms  $\widehat{\xi_i} \cong T/V(\xi_i)$  and  $\widehat{\xi} \cong T/V(\xi) = T/(\cap_i V(\xi_i))$ , the morphism  $\hat{\sigma}$  gets identified with the morphism  $\delta: T/(\cap_i V(\xi_i)) \rightarrow T/V(\xi_1) \times \dots \times T/V(\xi_k)$  induced by the diagonal map  $\Delta: T \rightarrow T^k$ .*

Next, let  $\gamma: \xi_1 \times \cdots \times \xi_k \rightarrow H \times \cdots \times H$  be the product of the inclusion maps  $\gamma_i: \xi_i \hookrightarrow H$ , and consider the induced homomorphism on character groups,

$$(25) \quad \hat{\gamma}: \widehat{H} \times \cdots \times \widehat{H} \longrightarrow \widehat{\xi_1} \times \cdots \times \widehat{\xi_k}.$$

The next lemma is immediate.

**Lemma 5.2.** *Under the isomorphisms  $\widehat{\xi_i} \cong T/V(\xi_i)$ , the morphism  $\hat{\gamma}$  gets identified with the projection map  $\pi: T^k \rightarrow T/V(\xi_1) \times \cdots \times T/V(\xi_k)$ .*

**5.2. Translated algebraic subgroups.** Given an algebraic subgroup  $P \subseteq T$ , and an element  $\eta \in T$ , denote by  $\eta P$  the translate of  $P$  by  $\eta$ . In particular, if  $C$  is an algebraic subtorus of  $T$ , then  $\eta C$  is a translated subtorus. If  $\eta$  is a torsion element of  $T$ , we denote its order by  $\text{ord}(\eta)$ . Finally, if  $A$  is a finite group, denote by  $c(A)$  the largest order of any element in  $A$ .

We are now in a position to state and prove the main result of this section (Theorem A from the Introduction). As before, let  $\xi_1, \dots, \xi_k$  be subgroups of  $H = \check{T}$ . Let  $\eta_1, \dots, \eta_k$  be elements in  $T$ , and consider the translated subgroups  $Q_1, \dots, Q_k$  of  $T$  defined by

$$(26) \quad Q_i = \eta_i V(\xi_i).$$

Clearly, each  $Q_i$  is a subvariety of  $T$ , but, unless  $\eta_i \in V(\xi_i)$ , it is not an algebraic subgroup.

**Theorem 5.3.** *Set  $\xi = \xi_1 + \cdots + \xi_k$  and  $\eta = (\eta_1, \dots, \eta_k) \in T^k$ . Then*

$$(27) \quad Q_1 \cap \cdots \cap Q_k = \begin{cases} \emptyset & \text{if } \hat{\gamma}(\eta) \notin \text{im}(\hat{\sigma}), \\ \rho V(\xi) & \text{otherwise,} \end{cases}$$

where  $\rho$  is any element in the intersection  $Q = Q_1 \cap \cdots \cap Q_k$ . Furthermore, if the intersection is non-empty, then

- (1) *The variety  $Q$  decomposes into irreducible components as  $Q = \bigcup_{\tau \in \widehat{\xi}/\xi} \rho \tau V(\bar{\xi})$ , and  $\dim(Q) = \dim(T) - \text{rank}(\xi)$ .*
- (2) *If  $\eta$  has finite order, then  $\rho$  can be chosen to have finite order, too. Moreover,  $\text{ord}(\eta) \mid \text{ord}(\rho) \mid \text{ord}(\eta) \cdot c(\bar{\xi}/\xi)$ .*

*Proof.* We have:

$$\begin{aligned} Q_1 \cap \cdots \cap Q_k \neq \emptyset &\iff \eta_1 a_1 = \cdots = \eta_k a_k, \text{ for some } a_i \in V(\xi_i) \\ &\iff \hat{\gamma}_1(\eta_1) = \cdots = \hat{\gamma}_k(\eta_k) && \text{by Lemma 5.2} \\ &\iff \hat{\gamma}(\eta) \in \text{im}(\hat{\sigma}) && \text{by Lemma 5.1.} \end{aligned}$$

Now suppose  $Q_1 \cap \cdots \cap Q_k \neq \emptyset$ . For any  $\rho \in Q_1 \cap \cdots \cap Q_k$ , and any  $1 \leq i \leq k$ , there is a  $\rho_i \in V(\xi_i)$  such that  $\rho = \eta_i \rho_i$ ; thus,  $\rho^{-1} \eta_i \in V(\xi_i)$ . Therefore,

$$\begin{aligned} \rho^{-1}(Q_1 \cap \cdots \cap Q_k) &= \rho^{-1}(\eta_1 V(\xi_1) \cap \cdots \cap \eta_k V(\xi_k)) \\ &= \rho^{-1}(\eta_1 V(\xi_1)) \cap \cdots \cap \rho^{-1}(\eta_k V(\xi_k)) \\ &= V(\xi_1) \cap \cdots \cap V(\xi_k) \\ &= V(\xi_1 + \cdots + \xi_k). \end{aligned}$$

Hence,  $Q_1 \cap \cdots \cap Q_k = \rho V(\xi)$ .

Finally, suppose  $\eta$  has finite order. Let  $\bar{\rho}$  be an element in  $T/V(\xi)$  such that  $\hat{\sigma}(\bar{\rho}) = \hat{\gamma}(\eta)$ . Note that  $\text{ord}(\bar{\rho}) = \text{ord}(\eta)$ . Using the exact sequence (12) and the third isomorphism theorem for groups, we get a short exact sequence,

$$(28) \quad 0 \longrightarrow \widehat{\xi/\xi} \longrightarrow T/V(\bar{\xi}) \xrightarrow{q} T/V(\xi) \longrightarrow 0.$$

Applying the  $\text{Hom}_{\text{group}}(-, \mathbb{C}^*)$  functor to the split exact (9), we get a split exact sequence,

$$(29) \quad 0 \longrightarrow V(\bar{\xi}) \longrightarrow T \xrightarrow{\begin{smallmatrix} s \\ \swarrow \quad \searrow \end{smallmatrix}} T/V(\bar{\xi}) \longrightarrow 0.$$

Now pick an element  $\tilde{\rho} \in q^{-1}(\bar{\rho})$ . We then have  $q(\tilde{\rho}^{\text{ord}(\eta)}) = \bar{\rho}^{\text{ord}(\eta)} = 1$ , which implies that  $\tilde{\rho}^{\text{ord}(\eta)} \in \widehat{\xi/\xi}$ . Hence,  $\tilde{\rho}$  has finite order in  $T/V(\bar{\xi})$ , and, moreover,  $\text{ord}(\eta) \mid \text{ord}(\tilde{\rho}) \mid \text{ord}(\tilde{\rho}) \cdot c(\bar{\xi}/\xi)$ . Setting  $\rho = s(\tilde{\rho})$  gives the desired translation factor.  $\square$

When  $k = 2$ , the theorem takes a slightly simpler form.

**Corollary 5.4.** *Let  $\xi_1$  and  $\xi_2$  be two subgroups of  $H$ , and let  $\eta_1$  and  $\eta_2$  be two elements in  $T = \widehat{H}$ . Then*

- (1) *The variety  $Q = \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2)$  is non-empty if and only if  $\eta_1 \eta_2^{-1}$  belongs to the subgroup  $V(\xi_1) \cdot V(\xi_2)$ .*
- (2) *If the above condition is satisfied, then  $\dim Q = \text{rank } H - \text{rank}(\xi_1 + \xi_2)$ .*

In the special case when  $H = \mathbb{Z}^r$  and  $T = (\mathbb{C}^*)^r$ , Theorem 5.3 allows us to recover Proposition 3.6 from [3].

**Corollary 5.5** (Hironaka [3]). *Let  $\xi_1, \dots, \xi_k$  be subgroups of  $\mathbb{Z}^r$ , let  $\eta = (\eta_1, \dots, \eta_k)$  be an element in  $(\mathbb{C}^*)^{rk}$ , and set  $Q_i = \eta_i V(\xi_i)$ . Then*

$$(30) \quad Q_1 \cap \dots \cap Q_k \neq \emptyset \iff \hat{\gamma}(\eta) \in \text{im}(\hat{\sigma}).$$

Moreover, for any connected component  $Q$  of  $Q_1 \cap \dots \cap Q_k$ , we have:

- (1)  $Q = \rho V(\bar{\xi})$ , for some  $\rho \in (\mathbb{C}^*)^r$ .
- (2)  $\dim(Q) = r - \text{rank}(\bar{\xi})$ .
- (3) If  $\eta$  has finite order, then  $\text{ord}(\eta) \mid \text{ord}(\rho) \mid \text{ord}(\eta) \cdot c(\bar{\xi}/\xi)$ .

**5.3. Some consequences.** We now derive a number of corollaries to Theorem 5.3. Fix a complex abelian reductive group  $T$ . To start with, we give a general description of the intersection of two arbitrary unions of translated subgroups.

**Corollary 5.6.** *Let  $W = \bigcup_i \eta_i V(\xi_i)$  and  $W' = \bigcup_j \eta'_j V(\xi'_j)$  be two unions of translated subgroups of  $T$ . Then*

$$(31) \quad W \cap W' = \bigcup_{i,j} \eta_i V(\xi_i) \cap \eta'_j V(\xi'_j),$$

where  $\eta_i V(\xi_i) \cap \eta'_j V(\xi'_j)$  is either empty (which this occurs precisely when  $\hat{\gamma}_{i,j}(\eta_i, \eta_j)$  does not belong to  $\text{im}(\hat{\sigma}_{i,j})$ ), or equals  $\eta_{i,j} V(\xi_i + \xi'_j)$ , for some  $\eta_{i,j} \in T$ .

**Corollary 5.7.** *With notation as above,  $W \cap W'$  is finite if and only if  $W \cap W' = \emptyset$  or  $\text{rank}(\xi_i + \xi'_j) = \dim(T)$ , for all  $i, j$ .*

**Corollary 5.8.** *Let  $W$  and  $W'$  be two unions of (torsion-) translated subgroups of  $T$ . Then  $W \cap W'$  is again a union of (torsion-) translated subgroups of  $T$ .*

The next two corollaries give a comparison between the intersections of various translates of two fixed algebraic subgroups of  $T$ . Both of these results will be useful in another paper [11].

**Corollary 5.9.** *Let  $T_1$  and  $T_2$  be two algebraic subgroups in  $T$ . Suppose  $\alpha, \beta, \eta$  are elements in  $T$ , such that  $\alpha T_1 \cap \eta T_2 \neq \emptyset$  and  $\beta T_1 \cap \eta T_2 \neq \emptyset$ . Then*

$$(32) \quad \dim(\alpha T_1 \cap \eta T_2) = \dim(\beta T_1 \cap \eta T_2).$$

*Proof.* Set  $\xi = \epsilon(T_1) + \epsilon(T_2)$ . From Theorem 5.3, we find that both  $\alpha T_1 \cap \eta T_2$  and  $\beta T_1 \cap \eta T_2$  have dimension equal to the corank of  $\xi$ . This ends the proof.  $\square$

**Corollary 5.10.** *Let  $T_1$  and  $T_2$  be two algebraic subgroups in  $T$ . Suppose  $\alpha_1$  and  $\alpha_2$  are torsion elements in  $T$ , of coprime order. Then*

$$(33) \quad T_1 \cap \alpha_2 T_2 = \emptyset \implies \alpha_1 T_1 \cap \alpha_2 T_2 = \emptyset.$$

*Proof.* Set  $H = \check{T}$ ,  $\xi_i = \epsilon(T_i)$ , and  $\xi = \xi_1 + \xi_2$ . By Theorem 5.3, the condition that  $T_1 \cap \alpha_2 T_2 = \emptyset$  implies  $\hat{\gamma}(1, \alpha_2) \notin \text{im}(\hat{\sigma})$ , where  $\sigma: \xi_1 \times \xi_2 \twoheadrightarrow \xi$  is the sum homomorphism, and  $\gamma: \xi_1 \times \xi_2 \hookrightarrow H \times H$  is the inclusion map.

Suppose  $\alpha_1 T_1 \cap \alpha_2 T_2 \neq \emptyset$ . Then, from Theorem 5.3 again, we know that  $\hat{\gamma}(\alpha_1, \alpha_2) \in \text{im}(\hat{\sigma})$ ; thus,  $(\hat{\gamma}(\alpha_1, \alpha_2))^n \in \text{im}(\hat{\sigma})$ , for any integer  $n$ . From our hypothesis, the orders of  $\alpha_1$  and  $\alpha_2$  are coprime; thus, there exist integers  $p$  and  $q$  such that  $p \text{ord}(\alpha_1) + q \text{ord}(\alpha_2) = 1$ . Hence,

$$\hat{\gamma}(1, \alpha_2) = \hat{\gamma}(\alpha_1^{p \text{ord}(\alpha_1)}, \alpha_2^{1-q \text{ord}(\alpha_2)}) = (\hat{\gamma}(\alpha_1, \alpha_2))^{p \text{ord}(\alpha_1)} \in \text{im}(\hat{\sigma}),$$

a contradiction.  $\square$

**5.4. Abelian covers.** In [11], we use Corollaries 5.9 and 5.10 to study the homological finiteness properties of abelian covers. Let us briefly mention one of the results we obtain as a consequence.

Let  $X$  be a connected CW-complex with finite 1-skeleton. Let  $H = H_1(X, \mathbb{Z})$  the first homology group. Since the space  $X$  has only finitely many 1-cells,  $H$  is a finitely generated abelian group. The characteristic varieties of  $X$  are certain Zariski closed subsets  $\mathcal{V}^i(X)$  inside the character torus  $\widehat{H} = \text{Hom}(H, \mathbb{C}^*)$ . The question we study in [11] is the following: Given a regular, free abelian cover  $X^\vee \rightarrow X$ , with  $\dim_{\mathbb{Q}} H_i(X^\vee, \mathbb{Q}) < \infty$  for all  $i \leq k$ , which regular, finite abelian covers of  $X^\vee$  have the same homological finiteness property?

**Proposition 5.11** ([11]). *Let  $A$  be a finite abelian group, of order  $e$ . Suppose the characteristic variety  $\mathcal{V}^i(X)$  decomposes as  $\bigcup_j \rho_j T_j$ , with each  $T_j$  an algebraic subgroup of  $\widehat{H}$ , and each  $\rho_j$  an element in  $\widehat{H}$  such that  $\bar{\rho}_j \in \widehat{H}/T_j$  satisfies  $\gcd(\text{ord}(\bar{\rho}_j), e) = 1$ . Then, given any regular, free abelian cover  $X^\vee$  with finite Betti numbers up to some degree  $k \geq 1$ , the regular  $A$ -covers of  $X^\vee$  have the same finiteness property.*



## 6. EXPONENTIAL INTERPRETATION

In this section, we explore the relationship between the correspondence  $H \rightsquigarrow T = \widehat{H}$  from §2 and the exponential map  $\text{Lie}(T) \rightarrow T$ .

**6.1. The exponential map.** Let  $T$  be a complex abelian reductive group. Denote by  $\text{Lie}(T)$  the Lie algebra of  $T$ . The exponential map  $\exp: \text{Lie}(T) \rightarrow T$  is an analytic map, whose image is  $T_0$ , the identity component of  $T$ .

As usual, set  $H = \check{T}$ , and consider the lattice  $\mathcal{H} = H^\vee \cong H/\text{Tors}(H)$ . We then can identify  $T_0 = \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*)$  and  $\text{Lie}(T) = \text{Hom}(\mathcal{H}^\vee, \mathbb{C})$ . Under these identifications, the corestriction to the image of the exponential map can be written as

$$(34) \quad \exp = \text{Hom}(-, e^{2\pi i z}): \text{Hom}(\mathcal{H}^\vee, \mathbb{C}) \rightarrow \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*),$$

where  $\mathbb{C} \rightarrow \mathbb{C}^*$ ,  $z \mapsto e^z$  is the usual complex exponential. Finally, upon identifying  $\text{Hom}(\mathcal{H}^\vee, \mathbb{C})$  with  $\mathcal{H} \otimes \mathbb{C}$ , we see that  $T_0 = \exp(\mathcal{H} \otimes \mathbb{C})$ .

The correspondence  $T \rightsquigarrow \mathcal{H} = (\check{T})^\vee$  sends an algebraic subgroup  $W$  inside  $T$  to the sublattice  $\chi = (\check{W})^\vee$  inside  $\mathcal{H}$ . Clearly,  $\chi = \text{Lie}(W) \cap \mathcal{H}$  is a primitive lattice; furthermore,  $\exp(\chi \otimes \mathbb{C}) = W_0$ .

Now let  $\chi_1$  and  $\chi_2$  be two sublattices in  $\mathcal{H}$ . Since the exponential map is a group homomorphism, we have the following equality:

$$(35) \quad \exp((\chi_1 + \chi_2) \otimes \mathbb{C}) = \exp(\chi_1 \otimes \mathbb{C}) \cdot \exp(\chi_2 \otimes \mathbb{C}).$$

On the other hand, the intersection of the two algebraic subgroups  $\exp(\chi_1 \otimes \mathbb{C})$  and  $\exp(\chi_2 \otimes \mathbb{C})$  need not be connected, so it cannot be expressed solely in terms of the exponential map. Nevertheless, we will give a precise formula for this intersection in Theorem 6.3 below.

**6.2. Exponential map and Pontrjagin duality.** First, we need to study the relationship between the exponential map and the correspondence from Proposition 2.1.

**Lemma 6.1.** *Let  $T$  be a complex abelian reductive group, and let  $\mathcal{H} = (\check{T})^\vee$ . Let  $\chi \leq \mathcal{H}$  be a sublattice. We then have an equality of connected algebraic subgroups,*

$$(36) \quad V((\mathcal{H}/\chi)^\vee) = \exp(\chi \otimes \mathbb{C}),$$

inside  $T_0 = \exp(\mathcal{H} \otimes \mathbb{C})$ .

*Proof.* Let  $\pi: \mathcal{H} \rightarrow \mathcal{H}/\chi$  be the canonical projection, and let  $K = \text{coker}(\pi^\vee)$ . As in (14), we have an exact sequence,  $0 \rightarrow K \rightarrow \chi^\vee \rightarrow \overline{\chi}/\chi \rightarrow 0$ . Applying the functor  $\text{Hom}(-, \mathbb{C}^*)$  to this sequence, we obtain a new short exact sequence,

$$(37) \quad 0 \longrightarrow \text{Hom}(\overline{\chi}/\chi, \mathbb{C}^*) \longrightarrow \text{Hom}(\chi^\vee, \mathbb{C}^*) \longrightarrow \text{Hom}(K, \mathbb{C}^*) \longrightarrow 0.$$

From the way the functor  $V$  was defined in (5), we have that  $\text{Hom}(K, \mathbb{C}^*) = V((\mathcal{H}/\chi)^\vee)$ . Composing with the map  $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$ , we obtain the following commutative diagram:

$$(38) \quad \begin{array}{ccc} \text{Hom}(\chi^\vee, \mathbb{C}) & \xrightarrow{\exp} & \text{Hom}(\chi^\vee, \mathbb{C}^*) \\ \downarrow \cong & & \downarrow \\ \text{Hom}(K, \mathbb{C}) & \xrightarrow{\exp} & \text{Hom}(K, \mathbb{C}^*) = V((\mathcal{H}/\chi)^\vee) \\ \downarrow & & \downarrow \\ \text{Hom}(\mathcal{H}^\vee, \mathbb{C}) & \xrightarrow{\exp} & \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*) \end{array}$$

Identify now  $\exp(\chi \otimes \mathbb{C})$  with the image of  $\text{Hom}(\chi^\vee, \mathbb{C})$  in  $\text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*)$ . Clearly, this image coincides with the image of  $V((\mathcal{H}/\chi)^\vee)$  in  $T_0 = \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*)$ , and so we are done.  $\square$

**Corollary 6.2.** *Let  $H$  be a finitely generated abelian group, and let  $\xi \leq H$  be a subgroup. Consider the sublattice  $\chi = (H/\xi)^\vee$  inside  $\mathcal{H} = H^\vee$ . Then*

$$(39) \quad V(\bar{\xi}) = \exp(\chi \otimes \mathbb{C}).$$

*Proof.* Note that  $\bar{\xi} = (\mathcal{H}/\chi)^\vee$ , as subgroups of  $H/\text{Tors}(H) = \mathcal{H}^\vee$ . The desired equality follows at once from Lemma 6.1.  $\square$

**6.3. Exponentials and determinant groups.** We are now in a position to state and prove the main result of this section (Theorem B from the Introduction).

**Theorem 6.3.** *Let  $T$  be a complex abelian reductive group, and let  $\chi_1$  and  $\chi_2$  be two sublattices of  $\mathcal{H} = \check{T}^\vee$ .*

- (1) *Set  $\xi = (\mathcal{H}/\chi_1)^\vee + (\mathcal{H}/\chi_2)^\vee \leq \mathcal{H}^\vee$  and  $\chi = (\mathcal{H}^\vee/\bar{\xi})^\vee \leq \mathcal{H}$ . Then*

$$\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) = \widehat{\bar{\xi}/\xi} \cdot \exp(\chi \otimes \mathbb{C}),$$

*as algebraic subgroups of  $T_0$ . Moreover, the identity component of both these groups is  $V(\bar{\xi}) = \exp(\chi \otimes \mathbb{C})$ .*

- (2) *Now suppose  $\chi_1$  and  $\chi_2$  are primitive sublattices of  $\mathcal{H}$ , with  $\chi_1 \cap \chi_2 = 0$ . We then have an isomorphism of finite abelian groups,*

$$\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) \cong \overline{\chi_1 + \chi_2} / \chi_1 + \chi_2.$$

*Proof.* To prove part (1), note that

$$\begin{aligned} \exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) &= V((\mathcal{H}/\chi_1)^\vee) \cap V((\mathcal{H}/\chi_2)^\vee) && \text{by Lemma 6.1} \\ &= V((\mathcal{H}/\chi_1)^\vee + (\mathcal{H}/\chi_2)^\vee) && \text{by Lemma 2.6} \\ &= \widehat{\bar{\xi}/\xi} \cdot V(\bar{\xi}) && \text{by Lemma 3.1.} \end{aligned}$$

Finally, note that  $(\mathcal{H}/\chi)^\vee = \bar{\xi}$ ; thus,  $V(\bar{\xi}) = \exp(\chi \otimes \mathbb{C})$ , again by Lemma 6.1.

To prove part (2), note that  $\bar{\xi} = \mathcal{H}^\vee$ , since we are assuming  $\chi_1 \cap \chi_2 = 0$ . Hence,  $V(\bar{\xi}) = \{1\}$ . Since we are also assuming that the lattices  $\chi_1$  and  $\chi_2$  are primitive, Proposition 3.5 applies, giving that  $\bar{\xi}/\xi \cong \overline{\chi_1 + \chi_2} / \chi_1 + \chi_2$ . Using now part (1) finishes the proof.  $\square$

The next corollary follows at once from Theorem 6.3.

**Corollary 6.4.** *Let  $H$  be a finitely generated abelian group, and let  $\xi_1$  and  $\xi_2$  be two subgroups. Let  $\mathcal{H} = H^\vee$  be the dual lattice, and  $\chi_i = (H/\xi_i)^\vee$  the corresponding sublattices.*

(1) *Set  $\xi = \overline{\xi_1} + \overline{\xi_2}$ . Then:*

$$\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) = \widehat{\overline{\xi}/\xi} \cdot V(\overline{\xi}).$$

(2) *Now suppose  $\text{rank}(\xi_1 + \xi_2) = \text{rank}(H)$ . Then*

$$\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) \cong \overline{\chi_1 + \chi_2} / \chi_1 + \chi_2 \cong \overline{\xi} / \xi.$$

**6.4. Some applications.** Let us now consider the case when  $T = (\mathbb{C}^*)^r$ . In this case,  $\mathcal{H} = \mathbb{Z}^r$ , and the exponential map (34) can be written in coordinates as  $\exp: \mathbb{C}^r \rightarrow (\mathbb{C}^*)^r$ ,  $(z_1, \dots, z_r) \mapsto (e^{2\pi i z_1}, \dots, e^{2\pi i z_r})$ . Applying Theorem 6.3 to this situation, we recover Theorem 1.1 from [6].

**Corollary 6.5** (Nazir [6]). *Suppose  $\chi_1$  and  $\chi_2$  are primitive lattices in  $\mathbb{Z}^r$ , and  $\chi_1 \cap \chi_2 = 0$ . Then  $\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C})$  is isomorphic to  $\overline{\chi_1 + \chi_2} / \chi_1 + \chi_2$ .*

As another application, let us show that intersections of subtori can be at least as complicated as arbitrary finite abelian groups.

**Corollary 6.6.** *For any integer  $n \geq 0$ , and any finite abelian group  $A$ , there exist subtori  $T_1$  and  $T_2$  in some complex algebraic torus  $(\mathbb{C}^*)^r$  such that  $T_1 \cap T_2 \cong (\mathbb{C}^*)^n \times A$ .*

*Proof.* Write  $A = \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_k}$ . Let  $\chi_1$  and  $\chi_2$  be the lattices in  $\mathbb{Z}^{2k}$  spanned by the columns of the matrices  $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ D \end{pmatrix}$ , where  $D = \text{diag}(d_1, \dots, d_k)$ . Clearly, both  $\chi_1$  and  $\chi_2$  are primitive, and  $\chi_1 \cap \chi_2 = 0$ , yet the lattice  $\chi = \chi_1 + \chi_2$  is *not* primitive if the group  $A = \overline{\chi} / \chi$  is non-trivial.

Now consider the subtori  $P_i = \exp(\chi_i \otimes \mathbb{C})$  in  $(\mathbb{C}^*)^{2k}$ , for  $i = 1, 2$ . By Corollary 6.5, we have that  $P_1 \cap P_2 \cong A$ . Finally, consider the subtori  $T_i = (\mathbb{C}^*)^n \times P_i$  in  $(\mathbb{C}^*)^{n+2k}$ . Clearly,  $T_1 \cap T_2 \cong (\mathbb{C}^*)^n \times A$ , and we are done.  $\square$

In particular, if  $A$  is a finite cyclic group, there exist 1-dimensional subtori,  $T_1$  and  $T_2$ , in  $(\mathbb{C}^*)^2$  such that  $T_1 \cap T_2 \cong A$ .

## 7. INTERSECTIONS OF TORSION-TRANSLATED SUBTORI

In this section, we revisit Theorem 5.3 from the exponential point of view. In the case when the translation factors have finite order, the criterion from that theorem can be refined, to take into account certain arithmetic information about the translated tori in question.

**7.1. Virtual belonging.** We start with a definition.

**Definition 7.1.** Let  $\chi$  be a primitive lattice in  $\mathbb{Z}^r$ . Given a vector  $\lambda \in \mathbb{Q}^r$ , we say  $\lambda$  *virtually belongs to  $\chi$*  if  $d \cdot \lambda \in \mathbb{Z}^r$ , where  $d = |\det[\chi \mid \chi_0]|$  and  $\chi_0 = \mathbb{Q}\lambda \cap \mathbb{Z}^r$ .

Here,  $[\chi \mid \chi_0]$  is the matrix obtained by concatenating basis vectors for the sublattices  $\chi$  and  $\chi_0$  of  $\mathbb{Z}^r$ . Note that  $\chi_0$  is a (cyclic) primitive lattice, generated by an element of the form  $\lambda_0 = m\lambda \in \mathbb{Z}^r$ , for some  $m \in \mathbb{N}$ .

In the next lemma, we record several properties of the notion introduced above.

**Lemma 7.2.** *With notation as above,*

- (1)  $\lambda$  virtually belongs to  $\chi$  if and only if  $d\lambda \in \chi_0$ .
- (2)  $d = 0$  if and only if  $\lambda \in \chi \otimes \mathbb{Q}$ , which in turn implies that  $\lambda$  virtually belongs to  $\chi$ .
- (3) If  $d > 0$ , then  $d$  equals the order of the determinant group  $\overline{\chi + \chi_0}/\chi + \chi_0$ .
- (4) If  $d = 1$ , then  $\lambda$  virtually belongs to  $\chi$  if and only if  $\lambda \in \mathbb{Z}^r$ , which happens if and only if  $\lambda \in \chi_0$ .

Next, we give a procedure for deciding when a torsion element in a complex algebraic torus belongs to a subtorus.

**Lemma 7.3.** *Let  $P \subset (\mathbb{C}^*)^r$  be a subtorus, and let  $\eta \in (\mathbb{C}^*)^r$  be an element of finite order. Write  $P = \exp(\chi \otimes \mathbb{C})$ , where  $\chi$  is a primitive lattice in  $\mathbb{Z}^r$ . Then  $\eta \in P$  if and only if  $\eta = \exp(2\pi i \lambda)$ , for some  $\lambda \in \mathbb{Q}^r$  which virtually belongs to  $\chi$ .*

*Proof.* Set  $n = \text{rank } \chi$ , and write  $\eta = \exp(2\pi i \lambda)$ , for some  $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Q}^r$ . Let  $\lambda_0$  be a generator of  $\chi_0 = \mathbb{Q}\lambda \cap \mathbb{Z}^r$ , and write  $\lambda = q\lambda_0$ . Finally, set  $d = |\det[\chi \mid \chi_0]|$ .

First suppose  $d = 0$ . Then, by Lemma 7.2(2),  $\lambda$  virtually belongs to  $\chi$ . Since  $\lambda \in \chi \otimes \mathbb{Q}$ , we also have  $\eta \in P$ , and the claim is established in this case.

Now suppose  $d \neq 0$ . As in the proof of Proposition 3.5, we can choose a basis for  $\mathbb{Z}^r$  so that the inclusion of  $\chi + \chi_0$  into  $\overline{\chi + \chi_0}$  has matrix of the form (15), with  $D = d$ . In this basis, the lattice  $\chi$  is the span of the first  $n$  coordinates, and  $\chi_0$  lies in the span of the first  $n + 1$  coordinates. Also in these coordinates,  $\lambda_{n+1} = q$ , and so  $\eta_{n+1} = e^{2\pi i q}$ ; furthermore,  $\eta_{n+2} = \dots = \eta_r = 1$ . On the other hand,  $P = \{z \in (\mathbb{C}^*)^r \mid z_{n+1} = \dots = z_r = 1\}$ . Therefore,  $\eta \in P$  if and only if  $dq$  is an integer, which is equivalent to  $d\lambda \in \chi_0$ .  $\square$

**7.2. Torsion-translated tori.** We are now ready to state and prove the main results of this section (Theorem C from the Introduction).

**Theorem 7.4.** *Let  $\xi_1$  and  $\xi_2$  be two sublattices in  $\mathbb{Z}^r$ . Set  $\varepsilon = \xi_1 \cap \xi_2$ , and write  $\widehat{\varepsilon}/\varepsilon = \{\exp(2\pi i \mu_k)\}_{k=1}^s$ . Also let  $\eta_1$  and  $\eta_2$  be two torsion elements in  $(\mathbb{C}^*)^r$ , and write  $\eta_j = \exp(2\pi i \lambda_j)$ . The following are equivalent:*

- (1) *The variety  $Q = \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2)$  is non-empty.*
- (2) *One of the vectors  $\lambda_1 - \lambda_2 - \mu_k$  virtually belongs to the lattice  $(\mathbb{Z}^r/\varepsilon)^\vee$ .*

*If either condition is satisfied, then  $Q = \rho V(\xi_1 + \xi_2)$ , for some  $\rho \in Q$ .*

*Proof.* Follows from Corollary 5.4 and Lemma 7.3.  $\square$

This theorem provides an efficient algorithm for checking whether two torsion-translated tori intersect. We conclude with an example illustrating this algorithm.

**Example 7.5.** Fix the standard basis  $e_1, e_2, e_3$  for  $\mathbb{Z}^3$ , and consider the primitive sublattices  $\xi_1 = \text{span}(e_1, e_2)$  and  $\xi_2 = \text{span}(e_1, e_3)$ . Then  $\varepsilon = \text{span}(e_1)$  is also primitive, and so  $s = 1$  and  $\mu_1 = 0$ . For selected values of  $\eta_1, \eta_2 \in (\mathbb{C}^*)^3$ , let us decide whether the set  $Q = \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2)$  is empty or not, using the above theorem.

First take  $\eta_1 = (1, 1, 1)$  and  $\eta_2 = (1, e^{2\pi i/3}, 1)$ , and pick  $\lambda_1 = 0$  and  $\lambda_2 = (1, \frac{1}{3}, 0)$ . One easily sees that  $d = \det \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = -3$ , and  $d(\lambda_1 - \lambda_2 - \mu_1) = (3, 1, 0) \in \mathbb{Z}^3$ . Thus,  $Q \neq \emptyset$ .

Next, take the same  $\eta_1$  and  $\lambda_1$ , but take  $\eta_2 = (e^{3\pi i/2}, 1, 1)$  and  $\lambda_2 = (\frac{3}{4}, 0, 1)$ . In this case,  $d = \det \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} = -3$  and  $d(\lambda_1 - \lambda_2 - \mu_1) = (\frac{9}{4}, 0, 3) \notin \mathbb{Z}^3$ . Thus,  $\mathcal{Q} = \emptyset$ .

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